$n\,(\mathfrak{G}_n \hookrightarrow \mathrm{Out}(\Pi_n))\}$

$)$

: inj and, the fact that "not depend prof" )

from $\mathrm{Out}^{FC}(\Pi_2^{tpd})$

$\longmapsto \tau_2$

from

of $\sigma$

$= \mathrm{Out}^{FC}(\Pi_n)^{cusp} \quad (n \geq 4)$

pf of (prop 1 ⇝ prop 2)

Consider the following comm diagram $\quad z^{log}$: geom gen fiber

$$
\begin{array}{ccccccccc}
 & & \pi_1(z_3^{log}) \simeq & & & & & & \\
1 & \longrightarrow & \Pi_{n+1/n-2} & \longrightarrow & \Pi_{n+1} & \longrightarrow & \Pi_{n-2} & \longrightarrow & 1 \\
 & & \pi_1(z_2^{log}) \simeq \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Pi_{n/n-2} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-2} & \longrightarrow & 1 \\
 & & \pi_1(z_1^{log}) \simeq \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Pi_{n-1/n-2} & \longrightarrow & \Pi_{n-1} & \longrightarrow & \Pi_{n-2} & \longrightarrow & 1
\end{array}
$$

Let $^{\forall}\sigma \in \mathrm{Out}^{FC}(\Pi_n)^{\Delta+}$ ; $\alpha \in \mathrm{Aut}^{FC}(\Pi_n)$ a lift of $\sigma$

Then $\alpha|_{\Pi_{n/n-2}} \in \mathrm{Aut}^{FC}(\Pi_{n/n-2})$ induces $\sigma_{n/n-2} \in \mathrm{Out}^{FC}(\Pi_{n/n-2})^{\Delta+}$

Thus, by Prop 1, there exist $^{\exists!}\tau \in \mathrm{Out}^{FC}(\Pi_{n+1/n-2})$

$$
\text{s.t.} \quad
\begin{array}{ccc}
\mathrm{Out}^{FC}(\Pi_{n+1/n-2}) & \longrightarrow & \mathrm{Out}^{FC}(\Pi_{n/n-2}) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\tau & \longmapsto & \sigma_{n/n-2}
\end{array}
$$

claim

$$\begin{tikzcd} \Pi_{n-2} \arrow[r] \arrow[d, "\mathrm{id}|_{\Pi_{n-2}}"', "\wr"] & \mathrm{Out}(\Pi_{n+1/n-2}) \arrow[d, "\wr", "\mathrm{Out}(\tau)"'] \\ \Pi_{n-2} \arrow[r] & \mathrm{Out}(\Pi_{n+1/n-2}) \end{tikzcd}$$

commutes

inj!

$$\begin{tikzcd} \Pi_{n-2} \arrow[r] \arrow[d, "\mathrm{id}|_{\Pi_{n-2}}"', "\wr"] & \mathrm{Out}^{FC}(\Pi_{n+1/n-2}) \arrow[r, hook] \arrow[d, "\wr", "\mathrm{Out}(\tau)"'] & \mathrm{Out}^{FC}(\Pi_{n/n-2}) \arrow[d, "\wr"', "\mathrm{Out}(\sigma_{n/n-2})"] \\ \Pi_{n-2} \arrow[r] & \mathrm{Out}^{FC}(\Pi_{n+1/n-2}) \arrow[r, hook] & \mathrm{Out}^{FC}(\Pi_{n/n-2}) \end{tikzcd}$$

//

In fact,

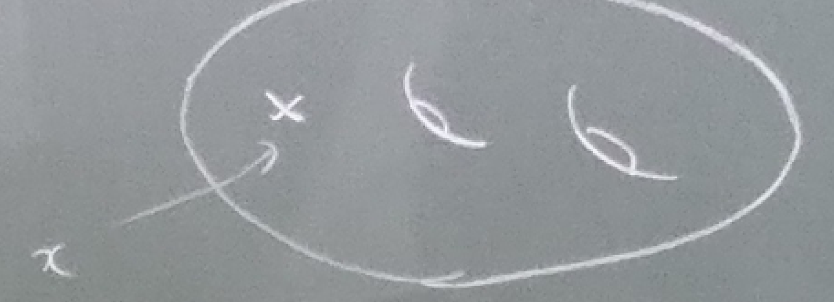
commutes

proof of Prop 1

$$\boxed{\mathrm{Im}\left(\mathrm{Out}^{FC}(\Pi_3) \longrightarrow \mathrm{Out}^{FC}(\Pi_2)\right) \supseteq \mathrm{Out}^{FC}(\Pi_2)^{\Delta +}}$$

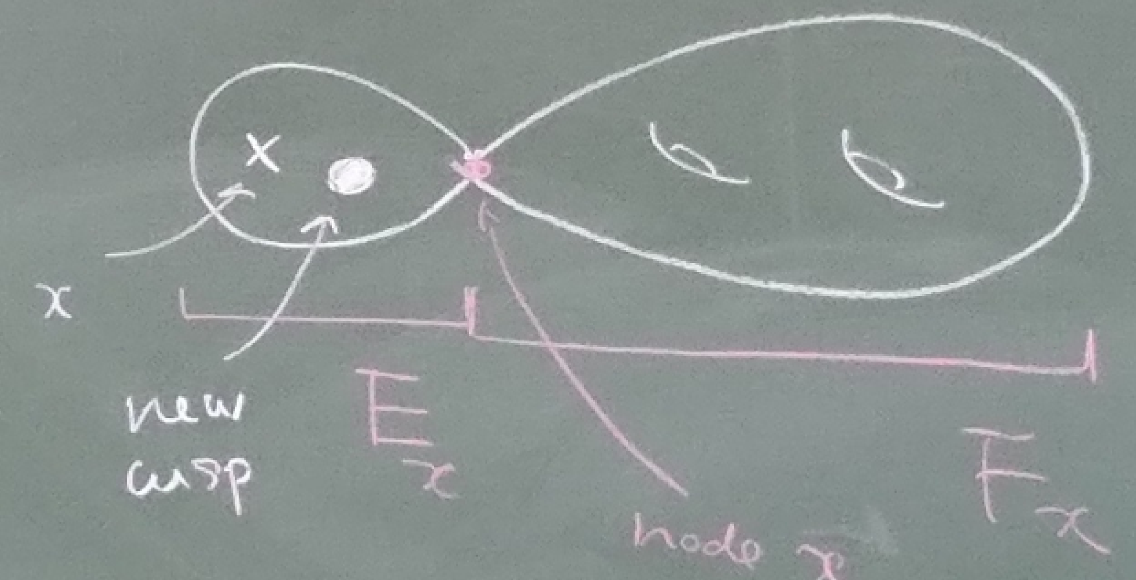
Def $X^{\log}$ : smooth log curve, $r \geq 1$

$x$ : cusp of $X^{\log}$

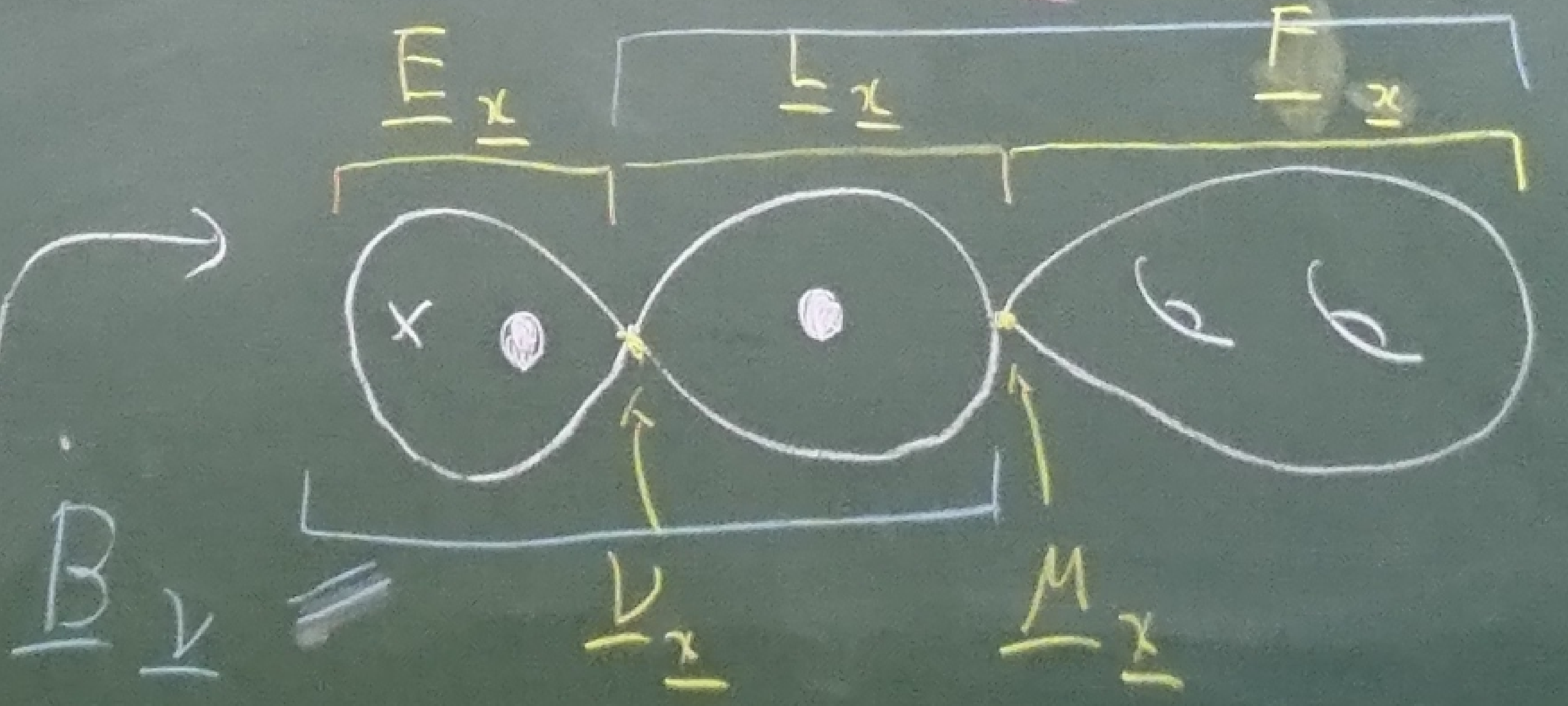



$$\begin{array}{ccc} X_2^{\log} & \longrightarrow & X^{\log} \\ \uparrow & \square & \uparrow \\ (X_2^{\log})_x & \longrightarrow & x \end{array}$$

forget "3"




$$\begin{array}{ccc} Pr_{12} : X_3^{\log} & \longrightarrow & X_2^{\log} \\ \uparrow & \square & \uparrow \\ (X_3^{\log})_x & \longrightarrow & x \end{array}$$




$$\boxed{\begin{aligned} B_\nu &:= E_x \cup L_x \\ B_\mu &:= L_x \cup F_x \end{aligned}}$$

Let $\forall \sigma \in \mathrm{Out}$

Then $\alpha \mid \Pi$

Thus, by Pr

s.t.

$\mathrm{Out}(\Pi_2)^{\Delta +}$

$\underline{B}_{\underline{M}}$

$\underline{B}_{\underline{v}} := \underline{E}_{\underline{x}} \cup \underline{L}_{\underline{x}}$

$\underline{B}_{\underline{M}} := \underline{L}_{\underline{x}} \cup \underline{F}_{\underline{x}}$

<u>Idea</u>

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \Pi_{3/2} \arrow[r, equal] \arrow[d] & \Pi_{3/2} \arrow[d] & & \\
1 \arrow[r] & \Pi_{3/1} \arrow[r] \arrow[d] & \Pi_3 \arrow[r] \arrow[d] & \Pi_1 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Pi_{2/1} \arrow[r] \arrow[d] & \Pi_2 \arrow[r] \arrow[d] & \Pi_1 \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

Take $\underline{\beta_2 \in \mathrm{Out}^{FC}(\Pi_2)^{\Delta +}}$

"$\Downarrow$"

$\beta_{2/1}|_{\Pi_{E_x}}$ $\beta_{2/1}|_{\Pi_{F_x}}$

"$\Downarrow$" since $\beta_2 \in \mathrm{Out}^{\Delta +}$

$\beta_{2/1}|_{\Pi_{E_x}}$ arises from $\exists\, \tau \in \mathrm{Out}(\Pi_{(E_x)_2})$

"$\Downarrow$"

Show "$\tau| = \beta_{2/1}|$"

$\Big\{$ "gluing"

$\exists\, \beta_{3/2} \in \mathrm{Out}^{FC}(\Pi_{3/2})$

compatibility of out action $\to$ "$\Downarrow$"

$\exists\, \beta_{3/1} \in \mathrm{Out}^{FC}(\Pi_{3/1})$

inj of comb cusp $\Big\}$

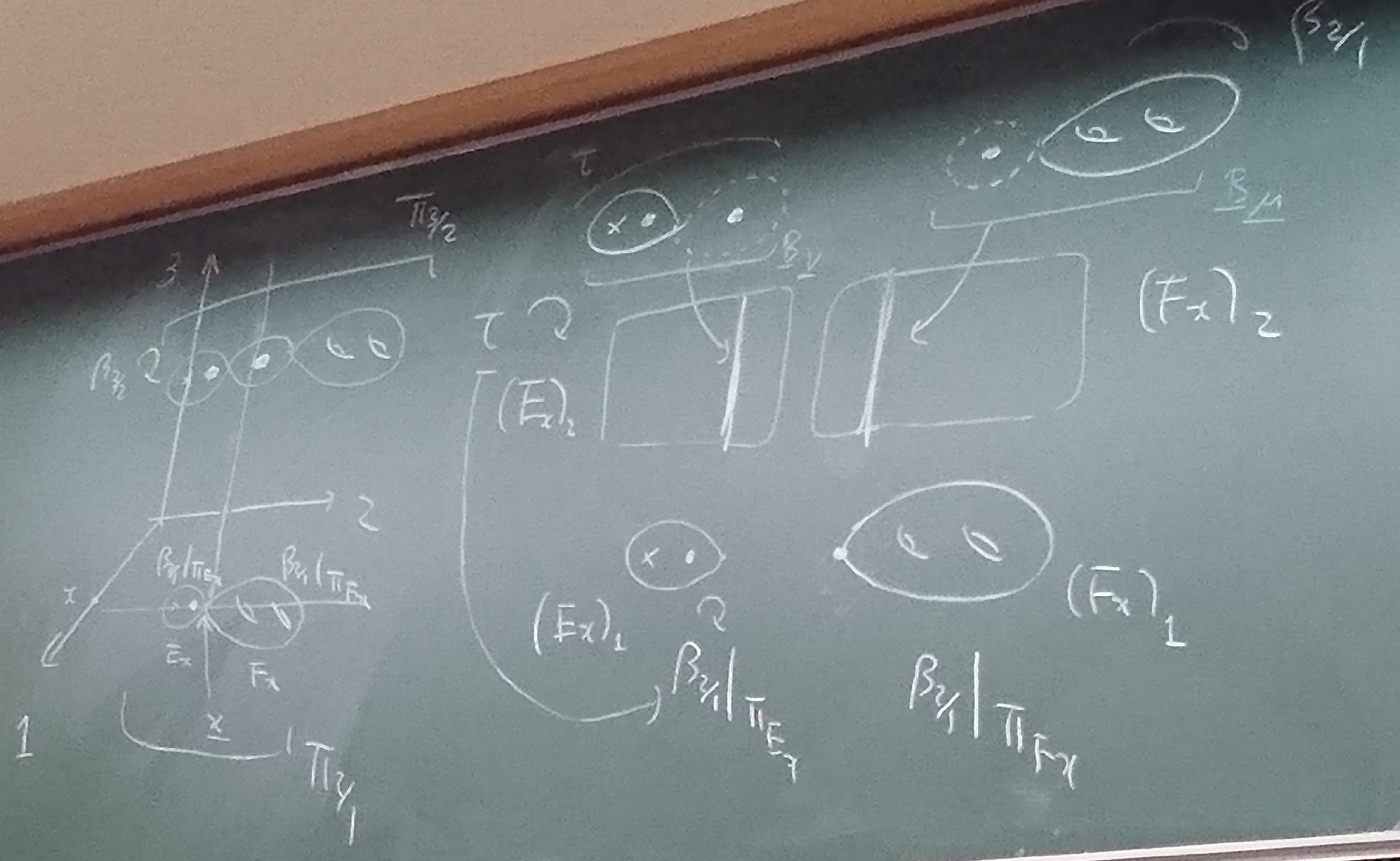
$\exists\, \beta_3 \in \mathrm{Out}^{FC}(\Pi_3)$

$\beta_{3/2}$ $\beta_{3/1}$ $\beta_{2/1}$

$1 \to \Pi_{3/2} \to \Pi_{3/1} \to \Pi_{2/1} \to 1$

$\beta_{3/1}$ $\beta_{3/2}$

$1 \to \Pi_{3/2} \to \Pi_3 \to \Pi_1 \to 1$




$X_2^{\log}$

forget "3" $\quad (X_2^{\log})_x$

$\mathrm{pr}_{12} : X_3^{\log}$

$(X_3^{\log})_{\underline{x}}$

proof (precise version)

$\beta_2 \in \mathrm{Out}^{Fc}(\Pi_2)^{\Delta +}$

$x$ : cusp of $X^{\log}$ s.t. $\beta_2 \in T_x^{-1}(GT)$

$\alpha_2 \in \mathrm{Aut}^{Fc}(\Pi_2)$ : a lift of $\beta_2$

$\alpha_1 \in \mathrm{Aut}^{Fc}(\Pi_1)$

S

Since $\beta_2 \in \mathrm{Out}^{FC}(\Pi_2)^{cusp}$, by replacing $\alpha_2$ by composite of $\alpha_2$ with $^{\exists}\Pi_2$-inn,

we may assume that $\alpha_1(I_x) = I_x$ ← inertia at $x$

Thus, by comb GC, $\alpha_2$ stabilizes $\Pi_{2/1}$-(nj) classes of $\Pi_{\underline{x}}$, $\Pi_{E_x}$, $\Pi_{F_x}$

Thus, by replacing $\alpha_2$ by comp of $\alpha_2$ with $^{\exists}\Pi_{2/1}$-inn,

we may assume that $\alpha_{2/1} := \alpha_2|_{\Pi_{2/1}}$ stabilizes $\Pi_{\underline{x}}$

$\Longrightarrow$ stabilizes the unique conjugates of $\Pi_{E_x}$, $\Pi_{F_x}$ which contain $\Pi_{\underline{x}}$

$x$

inertia at $x$

$\Pi_{2/1}$ -(conj) classes of

$\alpha_2$

$\Pi_{2/1}$ stabilizes $\Pi_{\underline{x}}$

gates of $\Pi_{E_x}$, $\Pi_{F_x}$

ich contain $\Pi_{\underline{x}}$

$\rightsquigarrow$ $\alpha_{2/1}|_{\Pi_{E_x}} \in \mathrm{Aut}^{FC}(\Pi_{E_x})$ , $\alpha_{2/1}|_{\Pi_{F_x}} \in \mathrm{Aut}^{FC}(\Pi_{F_x})$

determine $\begin{cases} \beta_{2/1} \in \mathrm{Out}^{FC}(\Pi_{2/1}) \\ \beta^{E}_{2/1} \in \mathrm{Out}^{FC}(\Pi_{E_x})^{\Delta +} \\ \beta^{F}_{2/1} \in \mathrm{Out}^{FC}(\Pi_{F_x}) \end{cases}$ $\Pi_{E_x} =: \Pi_1^{tpd}$

$\beta^{M}_{3/2} \in \mathrm{Out}^{FC}(\Pi_{\underline{B}_{\underline{M}}})$

$$\mathrm{Out}^{FCS}(\Pi_2^{tpd}) \longrightarrow \mathrm{Out}^{FC}(\Pi_1^{tpd})$$
$\Pi_1((E_x)_2)$

$$\exists!\, \beta_2^{tpd} \longmapsto \beta^{E}_{2/1} =: \beta_1^{tpd}$$

---

Note: we may restrict $\beta_2^{tpd} \in \mathrm{Out}^{FC}(\Pi_2^{tpd})$

to "$\beta_2^{tpd}|_{\Pi_{2/1}^{tpd}} \in \mathrm{Out}^{FC}(\Pi_{2/1}^{tpd})$" assoc to $\alpha_{2/1}|_{\Pi_{E_x}} \in \mathrm{Aut}^{FC}(\Pi_1^{tpd})$

$\Pi_{2/1}^{tpd} \cong \Pi_{\underline{B}_{\underline{v}}}$